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Dynamic behavior of cyclic automata networks ☆

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Abstract

We study the principal dynamical aspects of the cyclic automata on finite graphs.

We give bounds in the transient time and periodicity depending essentially on the graph structure. It is shown that there exist non-polynomial periods $e^{\Omega(\sqrt{|V|})}$, where $|V|$ denotes the number of sites in the graph.

To obtain these results we introduce some mathematical tools as continuity, firing paths, jumps and efficiency, which are interesting by themselves because they give a strong mathematical framework to study such discrete dynamical systems.

1. Introduction

Cyclic automata networks (CAN) consist of configurations of finite states, say $Q = \{0,\dots,q-1\}$, evolving synchronously in discrete time on an undirected finite connected graph $G=(V,E)$, where V is the set of sites and $E \subseteq V \times V$ is the set of edges. The local evolution rule is as follows: suppose the ith site is in state k. It advances to state $(k+1) \bmod q$ if and only if at least one neighbor has a copy of state $(k+1) \bmod q$. That is to say, the dynamics occurs in the set of all possible configurations of the graph over the finite set $Q = \{0,1,\dots,q-1\}$, $Q^V = \{x : V \to Q\}$. The transition function, $F : Q^V \to Q^V$, is given by

$$F_i(x) = \begin{cases} s(x_i) & \text{if } \exists j \in V_i \text{ such that } x_j = s(x_i), \\ x_i & \text{otherwise,} \end{cases}$$

where $F_i(x)$ denotes the ith component of $F(x)$, $s(p) = (p+1) \bmod q$ for $p \in Q$ and $V_i = \{j \in V : (i,j) \in E\}$ is the set of the neighbors of the site i.

This model belongs to the class of "excitable" automata [8]. A slightly related model is the Greenberg–Hasting automata, where the states change automatically from k to

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$(k+1) \bmod q$ whenever $k \neq 0$. For $k = 0$ the next state is 1 if and only if there exists some neighbor with this value.

Both models roughly described above have several applications: reaction–diffusion phenomena, oscillating chemical reactions, pattern formation [5, 6], sensory and cortical neural systems and simulations of the cardiac muscle [10]. More information about applications can be seen in [8].

When $q = 2$ a CAN constitutes a symmetric neural network. This case has been extensively studied and the basic results is that the cycles have period 1 or 2 and that it is possible to give sharp bounds for the transient time (see for instance [4]). In this paper we will consider the case $q \geqslant 3$ only.

From the mathematical point of view, the first model to be studied is the Greenberg–Hasting automaton [7]. In the previous reference, the authors introduce a discrete distance and an algebraic invariant which allows them to characterize persistent patterns in a regular two-dimensional lattice. Further, in [1], the pattern periodicity is characterized for two-dimensional lattices. On the other hand, in [10], Shangai studied a related two-dimensional finite model and characterized local rules such that their dynamics has bounded or non-bounded cycles in the dimension of the cellular space.

The CAN model has also been studied, from a probabilistic point of view, in a one-dimensional cellular space in the context of particle systems. Essentially, in [2, 3], authors give results about the fluctuation or fixation of the steady-state configuration starting from uniform product distributions.

In this work we give a bound for the transient time and the periodicity depending on the graph structure and on the number q, of states.

For each configuration $x \in Q^V$, we associate a subgraph of the original one composed by edges connecting sites whose states are neighbors in the state ring Q (the neighbors for k are $k+1$, $(k-1) \bmod q$ and k). This subgraph is called *the skeleton* associated to the configuration x. This notion was introduced by Allouche and Reder in [1] to study the dynamical behavior of the two-dimensional Greenberg–Hasting model.

We prove that the skeleton increases with the iteration. Since the skeleton is a subgraph of the original one, for time large enough the skeleton will be stable. We also see that the stable skeleton may be non-connected and we prove that the evolution of a site, say a, depends only on the connected component of the skeleton, containing a.

In this context we begin our study for configurations whose skeleton is the whole graph (continuous configurations), later, we study the general case by reducing the study to each connected component of the skeleton, which are continuous configurations.

We show that a necessary condition to insure that the period of the system is greater than 1, i.e. it is not a fixed point, is that the stable skeleton contains at least one circuit whose jump is not zero.

We also prove that the state at step t of a site i is the state at step 0 of the last site in a path beginning with i, with length smaller than t and maximum jump (this maximum is taken over every path beginning with i and having length smaller than t).

Furthermore, the length of previous path is the minimum over all the paths reaching the maximum jump.

In previous framework we prove that for continuous configurations the period divides the *lcm* of the lengths of the efficient circuits. The efficiency of a circuit is the quotient between its jump and its length. Furthermore, we are able to give a bound of the transient time.

Our study in the general case (i.e. non-necessarily continuous configurations) proceeds by giving a bound for the maximum time that a CAN could modify its skeleton and to reduce our study to continuous configurations.

We also build a family of graphs where the period is $4lcm\{i\colon i = 1,\dots,m\}$, which proves that our estimation for the global period is sharp. It is known that this last quantity is not polynomial in m, thus we exhibit a non-polynomial behavior for the CAN's period.

The paper is organized as follows: Section 2 is devoted to giving the basic definition; essentially the notions of *continuity* and *jump*. Also, we establish how these properties are related to the dynamics and we characterize the dynamical behavior for continuous configurations. Section 3 introduces, in our opinion, the main concept for a CAN: *the efficiency*. In this context we study how this concept is related to the notion of jump. Finally, in Section 4 we give the principal dynamical results for CAN, which can be summarized as follows:

- the characterization of fixed points for continuous configurations;
- the study of CAN's periods for the continuous and general case;
- the lower bounds for the transient length in the general case.

2. Dynamical behavior of a CAN for continuous configurations

In this section we prove that for a continuous configuration it is possible to know the state at any site i and at any time t by looking at paths in the graph G and computing their jumps at step 0.

We say that a state $x \in Q^V$ is *continuous* if and only if

$$\forall (i,j) \in E, \qquad x_i = x_j, \quad x_i = s(x_j) \quad \text{or} \quad s(x_i) = x_j.$$

We denote by $\mathscr{C}$ the set of all continuous configurations. Fig. 1 shows examples of non-continuous and continuous configurations.

Lemma 1. *Let $x \in \mathscr{C}$, then:*

(a) *$F(x)$ is also continuous.*

(b) *$\sigma(F_i(x), F_j(x)) = \sigma(x_i, x_j) + g_x(j) - g_x(i)$, for any $(i,j) \in E$, where*

$$g_x(i) = \begin{cases} 1 & \text{if } F_i(x) = s(x_i), \\ 0 & \text{otherwise}, \end{cases} \quad \text{and} \quad \sigma(x_i,x_j) = \begin{cases} 1 & s(x_i) = x_j, \\ 0 & x_i = x_j, \\ -1 & x_i = s(x_j). \end{cases}$$

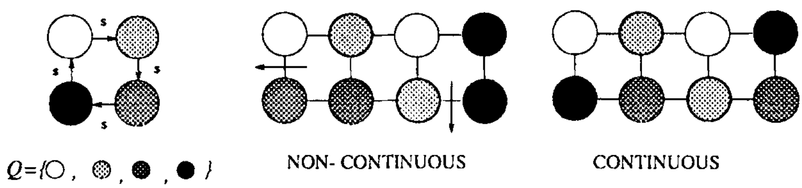


Fig. 1. Examples of non-continuous and continuous configurations.

(c) $g_x(i) = \max\{0, \sigma(x_i,x_j)\colon j \in V_i\}$.

Proof. Let $x \in \mathscr{C}$. Then, from definitions of σ and g we have that

$$\sigma(x_i,x_j) = -\sigma(x_j,x_i)$$

and

$$F_j(x) = s^{g_x(j)}(x_j) = s^{g_x(j)+\sigma(x_i,x_j)}(x_i) = s^{g_x(j)+\sigma(x_i,x_j)-g_x(i)}(F_i(x)).$$

Since $\sigma(x_i,x_j)=1$ implies that $g_x(i)=1$ and $\sigma(x_i,x_j)= -1$ implies that $g_x(j)=1$ we get

$$|\sigma(x_i,x_j) + g_x(j) - g_x(i)| \leqslant 1.$$

So, $F(x) \in \mathscr{C}$. Moreover, since $q \geqslant 3$ we obtain

$$\sigma(F_i(x),F_j(x)) = \sigma(x_i,x_j) + g_x(j) - g_x(i).$$

Now,

$$\max\{0,\sigma(x_i,x_j)\colon j \in V_i\} = 0 \;\Leftrightarrow\; \forall j \in V_i \;\sigma(x_i,x_j) \leqslant 0 \;\Leftrightarrow\; g_x(i) = 0. \qquad \square$$

We say that $C = (i_1,\ldots,i_n)$, $n \geqslant 2$ is a *path* of length $L(C) = n-1$ on the graph $G = (V,E)$ if and only if $i_j \in V$ and $(i_j,i_{j+1}) \in E$, for every $1 \leqslant j < n$.

Now, we define the *jump* of a path $C = (i_1,\ldots,i_n)$ over a continuous configuration x, as follows:

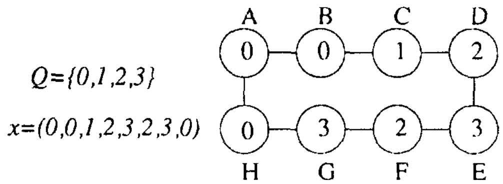
$$J(C,x) = \sum_{k=2}^{n} \sigma(x_{i_{k-1}},x_{i_k}).$$

It is easy to see that for a continuous configuration, $|J(C,x)| \leqslant L(C)$. Further, we have the following result.

Corollary 1 (Jump variation formula). *Let $C = (i_1,\ldots,i_n)$ be a path and $x \in \mathscr{C}$. Then,*

$$J(C,F(x)) = J(C,x) + g_x(i_n) - g_x(i_1).$$

Proof. Lemma 1(a) establishes that $F(x)$ is continuous, so $J(C,F(x))$ is well defined. From Lemma 1(b) we conclude. $\square$



$a(i,L,x)$

	1	2	3	4	5	6	7	8
A	0	1	2	3	3	3	4	4
B	1	2	3	3	3	4	4	4
C	1	2	2	2	3	3	3	4
D	1	1	1	2	2	2	3	4
E	-1	0	1	1	1	2	3	4
F	1	2	2	2	3	4	5	5
G	1	1	1	2	3	4	4	4
H	0	0	1	2	3	3	3	4

Fig. 2. Computation of $a(i,L,x)$ for $L \leqslant 8$, $Q = \{0,1,2,3\}$ and $x = (0,0,1,2,3,2,3,0)$.

We define the set of paths of length less or equal to L arising in site $i \in V$, as follows

$$G(i,L) = \{C = (i_1,\ldots,i_n)\colon i_1 = i,\ L(C) \leqslant L\}.$$

Also we define for $x \in \mathscr{C}$ the maximum jump on $G(i,L)$ by

$$a(i,L,x) = \max\{J(C,x)\colon C \in G(i,L)\}.$$

Fig. 2 gives an example where we compute $a(i,L,x)$.

It is easy to see that for $L \geqslant 1$, $a(i,L+1,x) \geqslant 0$. Indeed, let $j \in V_i$. We have two possibilities. First, $\sigma(x_i,x_j) < 0$ then the path $C = (i,j,i) \in G(i,2)$ has jump $J(C,x) = 0$. Second, $\sigma(x_i,x_j) \geqslant 0$ then $C = (i,j)$ belongs to $G(i,1)$ and its jump is $\sigma(x_i,x_j) \geqslant 0$.

Lemma 2. *For $x \in \mathscr{C}$ and $L \geqslant 1$, there exists $C^- \in G(i,L)$ such that*

$$J(C^-,F(x)) = a(i,L+1,x) - g_x(i).$$

Proof. Suppose that the maximum jump $a(i,L,x)$ is reached by a path $C=(i,\ldots,i_{n-1},i_n)$ of length $L(C) \geqslant 2$. Then, we can drop the last site in C obtaining a new path $C^- = (i,\ldots,i_{n-1})$ satisfying

$$J(C^-,x) = J(C,x) - \sigma(x_{i_{n-1}},x_{i_n}).$$

Using Corollary 1 to compute $J(C^-,F(x))$, in term of $J(C^-,x)$, we obtain

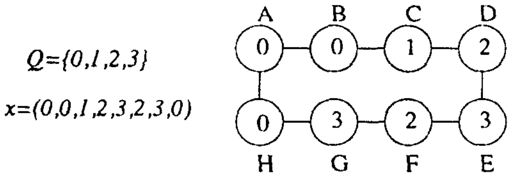
$$\begin{aligned} J(C^-,F(x)) &= J(C^-,x) + g_x(i_{n-1}) - g_x(i) \\ &= J(C,x) - g_x(i) - \sigma(x_{i_{n-1}},x_{i_n}) + g_x(i_{n-1}). \end{aligned}$$

We only need to prove that $\sigma(x_{i_{n-1}},x_{i_n}) = g_x(i_{n-1})$.

Since C reaches the maximum jump and $L(C) \geqslant 2$, we obtain $\sigma(x_{i_{n-1}},x_{i_n}) \geqslant 0$.

Let $j \in V_{i_{n-1}}$ and $C^j = (i,\ldots,i_{n-1},j)$. Then, $L(C^j) = L(C)$ and so $J(C^j,x) \leqslant J(C,x)$, which implies that $\sigma(x_{i_{n-1}},x_j) \leqslant \sigma(x_{i_{n-1}},x_{i_n})$ for every $j \in V_{i_{n-1}}$. So, from Lemma 1(c) we conclude that

$$g_x(i_{n-1}) = \max\{0, \sigma(x_{i_{n-1}},x_j) : j \in V_{i_{n-1}}\} = \sigma(x_{i_{n-1}},x_{i_n}).$$



$h(i,L,x)$

	1	2	3	4	5	6	7	8
A	1	2	3	4	4	4	7	7
B	1	2	3	3	3	6	6	6
C	1	2	2	2	5	5	5	8
D	1	1	1	4	4	4	7	8
E	1	2	3	3	3	6	7	8
F	1	2	2	2	5	6	7	7
G	1	1	1	4	5	6	6	6
H	1	1	3	4	5	5	5	8

Fig. 3. Computation of the minimal length.

Now, assume that the maximum jump is attained by a path $C = (i, i_n)$. Then, for every $j \in V_{i_n}$, the path $C^j = (i, i_n, j)$ belongs to $G(i,2) \subseteq G(i, L+1)$, for $L \geqslant 1$. Thus,

$$J(C^j,x) \leqslant J(C,x),$$

which implies that $\sigma(x_{i_n}, x_j) \leqslant 0$. Then, from Lemma 1(c) we have $g_x(i_n) = 0$. Taking $C^- = C$ and applying the Corollary 1 we get

$$J(C^-,F(x)) = J(C^-,x) + g_x(i_n) - g_x(i) = a(i,L+1,x) - g_x(i). \qquad \square$$

Lemma 3. *Let $x \in \mathscr{C}$ and $L \geqslant 1$. Then*

$$a(i,L,F(x)) + g_x(i) \leqslant a(i,L+1,x).$$

Proof. Let $C = (i, \ldots, i_n)$ be a path belonging to $G(i,L)$. We first prove that

$$J(C,x) + g_x(i_n) \leqslant a(i,L+1,x). \qquad (*)$$

Assume $g_x(i_n) = 0$. Then, previous inequality holds from the inclusion $G(i,L) \subseteq G(i, L+1)$. On the other hand, $g_x(i_n) = 1$ implies that $\sigma(x_{i_n}, x_j) = 1$ for some $j \in V_{i_n}$. We build a new path C^+, adding the edge (i_n, j) to C. Then, C^+ belongs to $G(i, L+1)$ and

$$J(C^+,x) = J(C,x) + \sigma(x_{i_n}, x_j) = J(C,x) + g_x(i_n),$$

which proves the inequality $(*)$.

Now, applying Corollary 1 to the left side of this inequality, we obtain that

$$J(C,F(x)) + g_x(i) = J(C,x) + g_x(i_n) \leqslant a(i,L+1,x).$$

Since C is an arbitrary element in $G(i,L)$ we conclude. $\square$

We define $h(i,L,x)$ as the minimum length over the paths in $G(i,L)$ which attain $a(i,L,x)$. Fig. 3 shows the values of $h(i,L,x)$ for the graph and the configuration given in Fig. 2.

Next proposition relates $h(i,L,x)$ to $a(i,L,x)$.

Proposition 1. *Let $x \in \mathscr{C}$. Suppose $L \geqslant 1$. Then*

$$a(i,L+1,x) = a(i,L,F(x)) + g_x(i).$$

Moreover:

(a) *if $h(i,L+1,x) \geqslant 2$, then*

$$a(i,L+1,x) > 0 \;\Rightarrow\; a(i,L,F(x)) > 0$$

and

$$h(i,L,F(x)) = h(i,L+1,x) - 1;$$

(b) *if $h(i,L+1,x) = 1$, then*

$$a(i,L,F(x)) = 0$$

and

$$h(i,L,F(x)) = 1.$$

Proof. Let $C = (i,\dots,i_{n-1},i_n) \in G(i,L)$ given in Lemma 2 satisfying

$$J(C,F(x)) = a(i,L+1,x) - g_x(i).$$

Lemma 3 establishes that

$$a(i,L,F(x)) \leqslant a(i,L+1,x) - g_x(i) = J(C,F(x)).$$

Thus,

$$a(i,L+1,x) = a(i,L,F(x)) + g_x(i).$$

(a) Suppose that $h(i,L+1,x) \geqslant 2$ and $a(i,L+1,x) > 0$. Then, if $g_x(i) = 1$ we obtain that $a(i,1,x) \geqslant 1$. Since $h(i,L+1,x) \geqslant 2$ we know $a(i,L+1,x) > 1$. Thus, from previous result, $a(i,L,F(x)) > 0$.

(b) Now, we analyze the case $h(i,L+1,x) = 1$. Let $C = (i,j)$ such that $J(C,x) = a(i,L+1,x)$, then $a(i,L+1,x) \in \{0,1\}$ and $g_x(i) = a(i,L+1,x)$. From Lemma 3 we get $a(i,L,F(x)) \leqslant 0$ and from Lemma 2 we know that $a(i,L,F(x)) \geqslant 0$. We conclude that $a(i,L,F(x)) = 0$. Moreover, from Corollary 1 we have

$$0 \geqslant J(C,F(x)) = J(C,x) + g_x(j) - g_x(i) \geqslant 0$$

then, we get $h(i,L,F(x)) = 1$. $\square$

Proposition 2. *Let $x \in \mathscr{C}$, then*

(a) *if $a(i,L,x) \leqslant 0$, then $F_i^l(x) = x_i$, $\forall l$, $1 \leqslant l \leqslant L$;*

(b) *if $a(i,L,x) > 0$, then $F_i^l(x) = s^{a(i,L,x)}(x_i)$, $\forall l$, $h(i,L,x) \leqslant l \leqslant L$.*

Proof. We prove the property (a) by induction on L.

For $L = 1$ and $C \in G(i,1)$ with $J(C,x) = a(i,L,x)$ we have

$$J(C,x) \leqslant 0 \iff \forall j \in V_i \sigma(x_i,x_j) \leqslant 0 \iff g_x(i) = 0 \iff F_i(x) = x_i.$$

Assume that the result holds for L', $1 \leqslant L' \leqslant L$. We prove it for $L+1$.

From Lemma 3, we obtain $a(i,L,F(x)) \leqslant a(i,L+1,x) \leqslant 0$. Applying the inductive hypothesis to $F(x)$ we obtain $F_i^l(F(x)) = F_i(x)$, $1 \leqslant l \leqslant L$. But $a(i,L+1,x) \leqslant 0$ implies that $\sigma(x_i,x_j) \leqslant 0$ for every $j \in V_i$, which is equivalent to $g_x(i) = 0$. Hence, $F_i(x) = x_i$ and the property (a) follows.

Now, we prove the property (b) by induction on L. For $L = 1$, let $C \in G(i,1)$ with $J(C,x) = a(i,1,x) > 0$, then $L(C) = J(C,x) = a(i,1,x) = h(i,1,x) = g_x(i) = 1$. Thus,

$$F_i(x) = s^{g_x(i)}(x_i) = s^{a(i,1,x)}(x_i).$$

Suppose that the property (b) holds for L', $1 \leqslant L' \leqslant L$. We prove the property (b) for $L+1$. From Proposition 1 we know $a(i,L,F(x)) > 0$ when $h(i,L+1,x) \geqslant 2$. So, in this case, we can apply the inductive hypothesis to $F(x)$ obtaining

$$F_i^l(F(x)) = s^{a(i,L,F(x))}(F_i(x)), \quad \forall l,\ h(i,L,F(x)) \leqslant l \leqslant L.$$

But, $h(i,L,F(x)) = h(i,L+1,x) - 1$ and $a(i,L,F(x)) = a(i,L+1,x) - g_x(i)$. So,

$$F_i^{l'}(x) = s^{a(i,L+1,x)-g_x(i)+g_x(i)}(x_i) = s^{a(i,L+1,x)}(x_i), \quad \forall l',\ h(i,L+1,x) \leqslant l' \leqslant L+1.$$

In the case $a(i,L+1,x) > 0$ and $h(i,L+1,x) = 1$, it follows from Proposition 1 that $a(i,L,F(x)) = 0$ and $h(i,L,F(x)) = 1$. Applying the property (a) to $F(x)$ we obtain

$$F_i^l(F(x)) = F_i(x) \quad \forall l,\ 1 \leqslant l \leqslant L.$$

But, $a(i,L+1,x) = 1$ and $h(i,L+1,x) = 1$, then

$$F_i(x) = s^{g_x(i)}(x_i) = s^{a(i,L+1,x)}(x_i);$$

hence,

$$F_i^{l'}(x) = s^{a(i,L+1,x)}(x_i) \quad \forall l',\ h(i,L+1,x) \leqslant l' \leqslant L+1. \qquad \square$$

Fig. 4 shows how to apply Proposition 2.

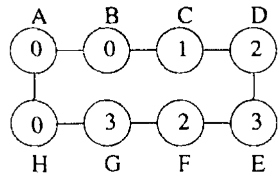
3. Efficiency and its relation with the maximum jump

In this section we give some relation between the maximum jump and the efficiency, which is the quotient between the jump and the length of a simple circuit. The following notions will be used throughout this section.

A path $C = (i_1,\ldots,i_n)$ is a *circuit* if and only if $i_1 = i_n$. Observe that with this definition, (i,j,i) is consider as a circuit, when $(i,j) \in E$. The path will be called *open* when C does not contain circuits. A circuit C is called *simple* if and only if every site in C appears only once.

$Q=\{0,1,2,3\}$

$x=(0,0,1,2,3,2,3,0)$



$h(i,L,x)$

	1	2	3	4	5	6	7	8
A	1	2	3	4	4	4	7	7
B	1	2	3	3	3	6	6	6
C	1	2	2	2	5	5	5	8
D	1	1	1	4	4	4	7	8
E	1	2	3	3	3	6	7	8
F	1	2	2	2	5	6	7	7
G	1	1	1	4	5	6	6	6
H	1	1	3	4	5	5	5	8

$a(i,L,x)$

	1	2	3	4	5	6	7	8
A	0	1	2	3	3	3	4	4
B	1	2	3	3	3	4	4	4
C	1	2	2	2	3	3	3	4
D	1	1	1	2	2	2	3	4
E	-1	0	1	1	1	2	3	4
F	1	2	2	2	3	4	5	5
G	1	1	1	2	3	4	4	4
H	0	0	1	2	3	3	3	4

$F^l(x)$

	0	1	2	3	4	5	6	7	8
A	0	0	1	2	3	3	3	0	0
B	0	1	2	3	3	3	0	0	0
C	1	2	3	3	3	0	0	0	1
D	2	3	3	3	0	0	0	1	2
E	3	3	3	0	0	0	1	2	3
F	2	3	0	0	0	1	2	3	3
G	3	0	0	0	1	2	3	3	3
H	0	0	0	1	2	3	3	3	0

Fig. 4. Computation of $F^l(x)$ using $a(i,L,x)$ and $h(i,L,x)$.

In Fig. 3 the path $C = (A,B,C,D,E,F,G,H,A,B,C,D,E,F,G,H,A)$ is a circuit and the path $C' = (A,B,C,D,E,F,G,H,A)$ is a simple one. Moreover, the path $C'' = (A,B,C,D)$ is an open path.

Let $\gamma = (i_1,\dots,i_n)$ be a simple circuit. It is easy to see that $J(\gamma,x) = -J(\gamma',x)$ where $\gamma' = (i_n,\dots,i_1)$. Then we must consider γ and γ' as two different circuits. Moreover, since $J(\gamma^\sigma,x) = J(\gamma,x)$ for any cyclic permutation σ, where $\gamma^\sigma = (i_{\sigma(1)},\dots,i_{\sigma(n)})$, we consider γ and γ^σ as the same circuit.

We use a description of a path C which allows us to compute $J(C,x)$ and $L(C)$ in terms of simple circuits (without vertex repetitions) and open paths of C (without circuits).

Let R be the number of simple circuits in the graph G. Consider the following algorithm.

input: *a path* C, *output*: *a path* $O(C)$ *and a vector* $n(C)$.

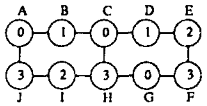
(1) Initially, set $O(C) = C$ and $n_r(C) = 0$, $r = 1,\dots,R$.

(2) Cover sites in C in the order described by C, looking for a simple circuit. If there is no simple circuit in $O(C)$ finish. Else, go to 3).

(3) If γ_r is such a simple circuit, add one to $n_r(C)$ and remove γ_r from $O(C)$. Back to step 2).

Clearly, at the end of previous algorithm $O(C)$ is an open path. Moreover, it is straightforward that

$$J(C,x) = J(O(C),x) + \sum_{r=1}^{R} n_r(C)J(\gamma_r,x)$$



G

$Q = \{0,1,2,3\}$

$x = (0,1,0,1,2,3,0,3,2,3)$

$R = 6$

(a)

$\gamma_1 = (C,D,E,F,G,H,C)$ $\gamma_4 = (C,H,G,F,E,D,C)$

$\gamma_2 = (A,B,C,H,I,J,A)$ $\gamma_5 = (A,J,I,H,C,B,A)$

$\gamma_3 = (A,B,C,D,E,F,G,H,I,J,A)$ $\gamma_6 = (A,J,I,H,G,F,E,D,C,B,A)$

$C = (A,J,I,H,C,D,E,F,G,H,C,B)$ $n(C) = (1,0,0,0,0,0)$

$O(C) = (A,J,I,H,C,B)$

	SIMPLE CIRCUIT			PATH
	γ_1	γ_2	γ_3	C
JUMP	4	0	4	5
EFFICIENCY	$\frac{4}{6}$	$\frac{0}{6}$	$\frac{2}{5}$	$\frac{4}{6}$
GLOBAL EFFICIENCY	$\frac{4}{6}$			

(b)

Fig. 5. Computation of efficiencies for a graph G with 6 simple circuits. (a) Connected graph G, continuous configuration x and $Q = \{0,1,2,3\}$. (b) Computation of efficiency for the simple circuits of G and for the path C.

and

$$L(C) = L(O(C)) + \sum_{r=1}^{R} n_r(C)L(\gamma_r),$$

where γ_r is the rth simple circuit. When $n_r(C) \neq 0$ we say that γ_r is *visited* or *used* by C.

Fig. 5 shows how we obtain $O(C)$ and $n(C)$ from a path C.

Now, we define the concept of efficiency and we use it to estimate the value of $a(i,L,x)$ and from that to determine the period and the transient length of the CAN.

We define the efficiency $e_r(x)$ for a simple circuit γ_r of G over a continuous configuration x by

$$e_r(x) = \frac{J(\gamma_r,x)}{L(\gamma_r)}.$$

For a path C visiting at least one simple circuit we define its efficiency over $x \in \mathscr{C}$ as

$$e(C,x) = \max\{e_r(x) : n_r(C) \neq 0\}.$$

The global efficiency $e(x)$ is defined by

$$e(x) = \max\{e_r(x) : r = 1,\dots,R\}.$$

Some easy properties of $e_r(x)$ and $e(x)$ are the following:

- $|e_r(x)| \leqslant 1$.
- Since $J(\gamma_r,x) = -J(\gamma_r',x)$ where $\gamma_r = (i_1,\dots,i_n)$ and $\gamma_r' = (i_n,\dots,i_1)$, and $J(\gamma_r,x) \leqslant L(\gamma_r)$ we have $0 \leqslant e(x) \leqslant 1$.

- Let $\gamma_r = (i_1,\dots,i_n)$ be a simple circuit. Then for $x \in \mathscr{C}$ we get $e_r(x) = qk/l$, $k,l \in \mathbb{N}$. In fact,

$$x_{i_1} = s^{\sigma(x_{i_1},x_{i_2})}(x_{i_2}) = \cdots = s^{J(\gamma_r,x)}(x_{i_1}).$$

thus $J(\gamma_r,x) = 0 \bmod q$. Moreover, when $J(\gamma_r,x) > 0$ we have $|V| \geqslant L(\gamma_r) \geqslant J(\gamma_r,x) \geqslant q$. Thus $e_r(x) \geqslant q/|V|$.

The next lemma proves that the maximum jump $a(i,\cdot,x)$ increases at least as a linear function on L with slope $e(x)$.

Lemma 4. *Let $x \in \mathscr{C}$ and $L \geqslant |V|$. Then for $i \in V$,*

$$a(i,L,x) \geqslant L \cdot e(x) - 2|V|.$$

Proof. When $e(x) = 0$ the result holds directly. Suppose $e(x) > 0$ and let r be such that the efficiency of γ_r is the global one. Let j_0 be a site in circuit γ_r such that its distance to i is minimal. If $i \neq j_0$, let v be an open path connecting i with j_0 and having minimum length. When $i = j_0$, we take v as the empty path. Since v is an open path and γ_r is a simple circuit we get

$$L(v) + L(\gamma_r) \leqslant |V|,$$

which implies $L(v) < L$. We build a path C, beginning with v and followed by m loops around circuit γ_r, where m is given by

$$m = \left\lfloor \frac{L - L(v)}{L(\gamma_r)} \right\rfloor .$$

Then $L(C) = L(v) + mL(\gamma_r) \leqslant L(v) + L - L(v) = L$, i.e., $C \in G(i,L)$. Moreover, since $L(v) \leqslant |V| - L(\gamma_r)$ we have that $m \geqslant \lfloor (L - |V|)/L(\gamma_r) + 1 \rfloor \geqslant (L - |V|)/L(\gamma_r)$ which implies

$$\begin{aligned} J(C,x) &= J(v,x) + m \cdot J(\gamma_r,x) \geqslant -L(v) + \frac{L - |V|}{L(\gamma_r)} J(\gamma_r,x) \\ &\geqslant -|V| + L \cdot e_r(x) - |V| e_r(x) \geqslant L \cdot e_r(x) - 2|V| = L \cdot e(x) - 2|V|. \end{aligned}$$

Since $a(i,L,x) \geqslant J(C,x)$ we conclude. □

Next lemma establishes that two simple circuits used in a path C in $G(i,L)$, with $a(i,L,x) = J(C,x)$ and having the same length, must have the same efficiency.

Lemma 5. *Consider $C \in G(i,L)$ with $J(C,x) = a(i,L,x)$ and two vertices j,k such that $n_j(C) \neq 0$ and $n_k(C) \neq 0$. Then, $L(\gamma_j) = L(\gamma_k)$ implies $J(\gamma_j,x) = J(\gamma_k,x)$.*

Proof. We build two paths C' and C'' as follows (see also Fig. 6).

$$C' : \quad O(C') = O(C),$$

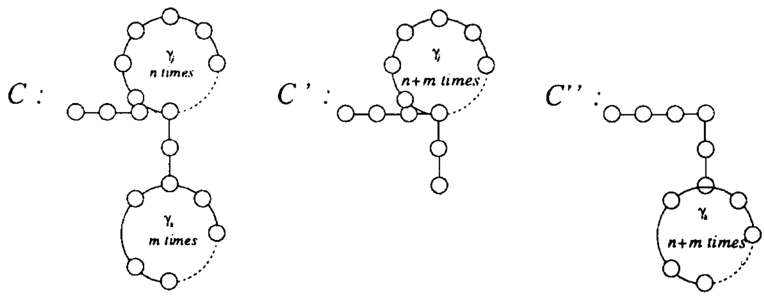


Fig. 6. Example of how we proceed in the proof of Lemma 5.

$$n_r(C') = \begin{cases} n_r(C), & r \neq j \text{ and } r \neq k, \\ n_j(C) + n_k(C), & r = k, \\ 0, & r = j, \end{cases}$$

$$C'': \quad O(C'') = O(C),$$

$$n_r(C'') = \begin{cases} n_r(C), & r \neq j \text{ and } r \neq k, \\ n_j(C) + n_k(C), & r = j, \\ 0, & r = k. \end{cases}$$

Since γ_j and γ_k are simple circuits C' and C'' are well defined. Moreover, $L(\gamma_j) = L(\gamma_k)$ implies $L(C) = L(C') = L(C'')$.

Since $C \in G(i,L)$ and $J(C,x) = a(i,L,x)$ we know $J(C',x) \leqslant J(C,x)$ and $J(C'',x) \leqslant J(C,x)$. Therefore,

$$(n_j(C) + n_k(C))J(\gamma_k,x) \leqslant n_j(C)J(\gamma_j,x) + n_k(C)J(\gamma_k,x)$$

and

$$(n_j(C) + n_k(C))J(\gamma_j,x) \leqslant n_j(C)J(\gamma_j,x) + n_k(C)J(\gamma_k,x).$$

Hence,

$$n_j(C)J(\gamma_k,x) \leqslant n_j(C)J(\gamma_j,x)$$

and

$$n_k(C)J(\gamma_j,x) \leqslant n_k(C)J(\gamma_k,x).$$

Since $n_j(C) \neq 0$ and $n_k(C) \neq 0$ we conclude $J(\gamma_j,x) = J(\gamma_k,x)$. $\square$

The previous result motivates us to separate the computation of the jump $J(C,x)$ in three parts. Let Ω, Ω_{ne} and Ω_{e} be the set of simple circuits visited by C, the set of circuits in Ω whose efficiency is less than $e(C,x)$ and the set of circuits in Ω whose efficiency is equal to $e(C,x)$, respectively. We call a simple circuit in Ω_{e} an efficient circuit for C and a simple circuit in Ω_{ne} a non-efficient circuit for C.

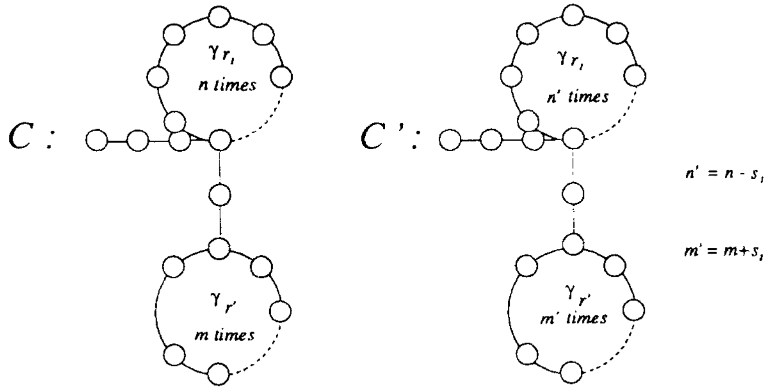


Fig. 7. Picture of how we proceed in Lemma 6.

Let $C \in G(i,L)$, with $J(C,x) = a(i,L,x)$. We prove that when a non-efficient circuit γ_r for C has length s, then C visits simple circuits with length s at most $|V|$ times. Notice that from Lemma 5 any simple circuit of length s visited by C is a non-efficient one.

Lemma 6. *Let $x \in \mathscr{C}$ and $C \in G(i,L)$ with $J(C,x) = a(i,L,x)$. Let r_1 be, such that γ_{r_1} is a non-efficient circuit for C. Then, for $s = L(\gamma_{r_1})$,*

$$a_s = \sum_{L(\gamma_r)=s}^{R} n_r(C) < |V|.$$

Proof. Suppose that $|V| \leqslant a_s$. Let $\gamma_{r'}$ be an efficient circuit for C. Define C' as follows (see Fig. 7).

$$C': \quad O(C') = O(C),$$

$$n_r(C') = \begin{cases} n_{r'}(C) + s, & r = r', \\ a_s - L(\gamma_{r'}), & r = r_1, \\ 0, & r \neq r_1 \text{ and } L(\gamma_r) = s, \\ n_r(C), & \text{otherwise.} \end{cases}$$

Since $a_s \geqslant |V| \geqslant L(\gamma_{r'})$ and from Lemma 5 $L(\gamma'_r) \neq s$, C' is well defined. Moreover, $L(C') - L(C)$ is given by

$$L(C') - L(C) = (n_{r'}(C) + s)L(\gamma_{r'}) + (a_s - L(\gamma_{r'}))s$$
$$-n_{r'}(C)L(\gamma_{r'}) - n_{r_1}(C)s - \sum_{L(\gamma_r)=s, r\neq r_1}^{R} n_r(C)L(\gamma_r).$$

Since $\sum_{L(\gamma_r)=s}^{R} n_r(C)L(\gamma_r) = sa_s$ we conclude $L(C') = L(C)$, which implies $C' \in G(i,L)$.

Now, we compute the jump difference between the two paths:

$$\begin{aligned}J(C',x)-J(C,x)=&(n_{r'}(C)+s)J(\gamma_{r'},x)+(a_s-L(\gamma_{r'}))J(\gamma_{r_1},x)\\&-n_{r'}(C)J(\gamma_{r'},x)-n_{r_1}(C)J(\gamma_{r_1},x)\\&-\sum_{L(\gamma_r)=s,r\neq r_1}^{R}n_r(C)J(\gamma_r,x).\end{aligned}$$

From Lemma 5 we know that $\sum_{L(\gamma_r)=s}^{R} n_r(C)J(\gamma_r,x)=J(\gamma_{r_1},x)a_s$. Then

$$J(C',x)-J(C,x)=sJ(\gamma_{r'},x)-L(\gamma_{r'})J(\gamma_{r_1},x)=sL(\gamma_{r'})(e_{r'}(x)-e_{r_1}(x))>0,$$

which is a contradiction, since $J(C,x)=a(i,L,x)$. □

We define the open part of $J(C,x)$ as $J(O(C),x)$, the non-efficient part as the contribution of the non-efficient circuits for C and the efficient part as the contribution of the efficient circuits for C.

In Corollary 2, we shall prove that the open part and the non-efficient part of the jump are bounded by a polynomial on $|V|$. So, the linear growth of $a(i,L,x)$ as a function on L established in Lemma 4, is due to the efficient part.

Corollary 2. *Let $x\in\mathscr{C}$ and $C\in G(i,L)$ with $a(i,L,x)=J(C,x)$. Then*:

(a) $J(O(C),x)\leqslant|V|$ *and* $\sum_{r\in\Omega_{\mathrm{ne}}}n_r(C)J(\gamma_r,x)\leqslant|V|^3$.

(b) $J(C,x)<e(C,x)L(C)|+|V|+|V|^3$.

Proof. Clearly, $J(O(C),x)\leqslant L(C)\leqslant|V|$.

From Lemma 6 we know that C visits non-efficient circuit of length s at most $|V|$ times. So, it visits non-efficient circuit at most $|V|^2$ times. Moreover, the jump of any simple circuit is bounded by V. Therefore the non-efficient part is bounded by $|V|^3$.

To obtain the property (b) we perform the following computation.

$$J(C,x)=J(O(C),x)+\sum_{r\in\Omega_{\mathrm{ne}}}n_r(C)J(\gamma_r,x)+\sum_{r\in\Omega_{\mathrm{e}}}n_r(C)J(\gamma_r,x).$$

It is clear that

$$\begin{aligned}\sum_{r\in\Omega_{\mathrm{e}}}n_r(C)J(\gamma_r,x)&=e(C,x)\sum_{r\in\Omega_{\mathrm{e}}}n_r(C)L(\gamma_r)\\&=e(C,x)(L(C)-\sum_{r\in\Omega_{\mathrm{ne}}}n_r(C)L(\gamma_r)-L(O(C))).\end{aligned}$$

Thus,

$$\begin{aligned}J(C,x)=&J(O(C),x)-e(C,x)L(O(C))+\sum_{r\in\Omega_{\mathrm{ne}}}n_r(C)(J(\gamma_r,x)\\&-e(C,x)L(\gamma_r))+e(C,x)L(C),\end{aligned}$$

since $-e(C,x)L(O(C)) \leqslant 0$, $J(O(C),x) \leqslant |V|$, $-e(C,x)L(\gamma_r) \leqslant 0$, $J(\gamma_r,x) \leqslant |V|$ and $\sum_{r\in\Omega_{\text{nc}}} n_r(C) \leqslant |V|^2$ we obtain the conclusion. $\square$

The previous corollary says us that $a(i,L,x)$ is bounded from above by a linear function on L with slope $e(C,x)$. Lemma 4 and Corollary 2 together give the next result.

Lemma 7. *Let $x \in \mathscr{C}$, $L \geqslant 4|V|^5/q$, $e(x) > 0$ and $C \in G(i,L)$ with $J(C,x) = a(i,L,x)$. Then*

$$e(C,x) = e(x).$$

Proof. It is clear that $L > 4|V|^4/q \geqslant |V|$ so, Lemma 4 implies

$$J(C,x) \geqslant Le(x) - 2|V|.$$

Moreover, from Corollary 2(b), we get

$$Le(x) - 2|V| \leqslant J(C,x) < e(C,x)L(C) + |V| + |V|^3 \leqslant e(C,x)L + |V| + |V|^3.$$

Assume that $e(C,x) = 0$. Then we get that $L \leqslant (3|V| + |V|^3)/e(x)$. Since $e(x) > 0$ we know that $e(x) \geqslant q/|V|$. Thus, $L \leqslant 4|V|^4/q$ which is a contradiction. So, $e(C,x) = kq/l$ with $k \in \mathbb{N}$, $k \neq 0$ and $l \leqslant |V|$. Suppose that $e(C,x) = kq/l < k'q/l' = e(x)$. Since $l, l' \leqslant |V|$ and $k'l - kl' \geqslant 1$ we get that

$$e(x) - e(C,x) = \frac{k'q}{l'} - \frac{kq}{l} = \frac{q}{ll'}(k'l - kl') \geqslant \frac{q}{|V|^2}.$$

So,

$$L(e(x) - e(C,x)) \leqslant 3|V| + |V|^3$$

implies that

$$L < \frac{4|V|^3}{e(x) - e(C,x)} \leqslant \frac{4|V|^3}{q}|V|^2,$$

which is a contradiction. $\square$

Let $C \in G(i,L)$ with $J(C,x) = a(i,L,x)$. Let s be such that there exists an efficient circuit for C of length s. Then, we choose one efficient circuit of length s, say us γ^s, and we accumulate all the loops performed by C in efficient circuits for C of length s, in γ^s.

The *accumulated form* of C, C_{a}, consists of the modification of the efficient part of the jump of C, by grouping, for each s, the loops made by C in efficient circuits with lengths s, in γ^s. No other efficient circuit for C of length s, is visited by C_{a}.

Since two efficient circuits for C, with the same length have the same jump, it is easy to see that $L(C_{\text{a}}) = L(C)$ and $J(C_{\text{a}},x) = J(C,x)$.

Lemma 8. *Let $x \in \mathscr{C}$, $e(x) > 0$ and r be such that γ_r has global efficiency and i a site in γ_r. Then, for $L \geqslant 3|V|^4/q$ there exists $C \in G(i,L)$, with $J(C,x) = a(i,L,x)$, visiting γ_r.*

Proof. Let $C \in G(i,L)$ with $J(C,x) = a(i,L,x)$. We know from Corollary 2(a) that the non-efficient part of C is bounded by $|V|^3$ and from Lemma 4 that the jump increases at least as a linear function of L.

Assume that for every s, the loops performed by C, in efficient circuits for C of length s, are bounded by $|V|$. Then, the efficient part of the jump will be bounded by $|V||V|^2$, hence $e(x) \cdot L - |V| \leqslant J(C,x) < 2|V|^3 + |V|$. Since $e(x) > 0$ we know that $e(x) \geqslant q/|V|$. Thus, $L < (2|V|^3 + |V|)/e(x) \leqslant 3|V|^4/q$, which is a contradiction. So, there exists s' such that $N > |V|$, where N is the number of loops performed by C in efficient circuits for C of length s'.

Let C_a be the accumulated form of C and $\{\gamma^s\}_{s \geqslant 3}$ the simple circuits visited by C_a. Let s' be as above. We define a new path which visits $\gamma^{s'}$ only $N - L(\gamma_r)$ times and γ_r exactly s' times. It is easy to see that the length and the jump of C and this new path agree. Thus we obtain the conclusion. $\square$

Let n_r be the loops performed by C_a in efficient circuit of length r. For a path C and a length s such that there exists an efficient circuit for C with length s, we define the *accumulated form of C and s*, as the path $C_{a,s}$ which is derived from C_a by concentrating loops in γ^s (see Fig. 8). Let $n_{s'}$ be the number of loops performed by C_a in $\gamma^{s'}$. $C_{a,s}$ visits at most one efficient circuit of length s' for $s' = 1, \ldots, |V|$. The number of loops performed in the efficient circuit of length s' is given by $n'_{s'} = n_{s'} \bmod s$ and the number of loops performed in the efficient circuit of length s is given by

$$n'_s = n_s + \sum_{s'=3, s' \neq s}^{|V|} s' \left\lfloor \frac{n_{s'}}{s} \right\rfloor .$$

It is not difficult to prove that $L(C_{a,s}) = L(C)$ and $J(C_{a,s},x) = J(C,x)$. In the next lemma we give an upper bound for n'_s.

Lemma 9. *Let $x \in \mathscr{C}$ and $C \in G(i,L)$ with $J(C,x) = a(i,L,x)$. Let $C_{a,s}$ be the accumulated form of C and s. Then $e(x) > 0$ and $L \geqslant 4|V|^4/q$ implies*

$$n'_s s \geqslant L - \frac{5|V|^4}{q}.$$

Proof. From Lemma 4 and Corollary 5(a) we know that the efficient part of $J(C,x)$ is bounded from below by $Le(x) - 3|V| - |V|^3$. Moreover, from the definition of $C_{a,s}$, we know that the efficient part of $J(C,x)$ and $J(C_{a,s},x)$ agree, which implies that the efficient part of $J(C,x)$ is given by

$$\sum_{s'=3}^{|V|} n'_{s'} s' e(C,x).$$

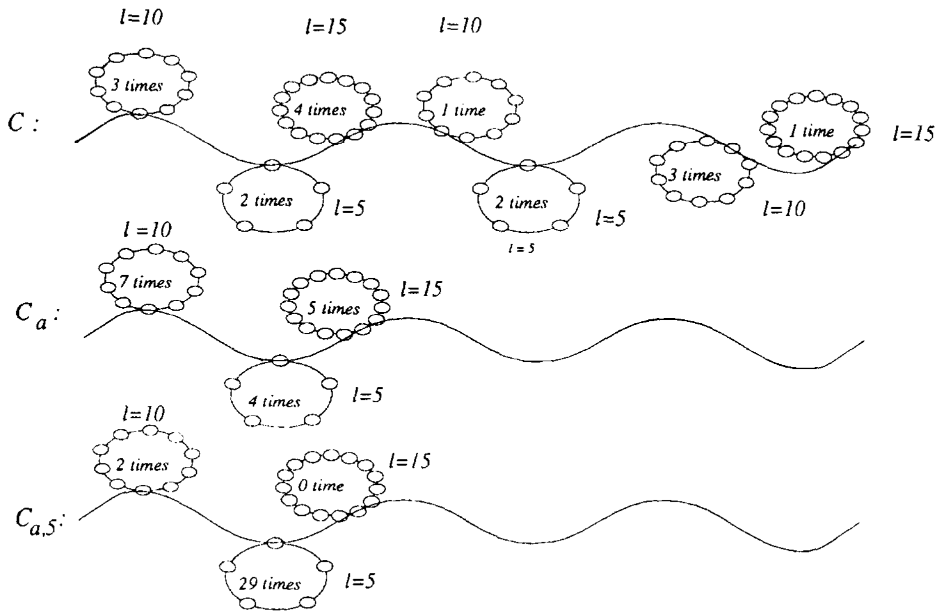


Fig. 8. Computation of C_a and $C_{a,s}$.

But, for $L \geqslant 4|V|^4/q$, Lemma 7 proves that the efficiency of C agree with the global efficiency. Hence,

$$e(x) \sum_{s'=3, s' \neq s}^{|V|} n'_{s'}s + e(x)n'_s s > Le(x) - 3|V| - |V|^3.$$

The first term in the equation above is bounded by $|V|^3$, so

$$n'_s s \geqslant L - \frac{3|V| + 2|V|^3}{e(x)} > L - \frac{5|V|^4}{q}. \qquad \square$$

4. Dynamical results for the CAN model

In this section we apply the results obtained throughout the paper, to give the most important dynamical results of the CAN model.

For a configuration x we define *the skeleton* $(V, S(x))$ as a subgraph of G, where $S(x)$ is given by

$$S(x) = \{(i,j) \in E : x_i = s(x_j) \vee x_i = x_j \vee s(x_i) = x_j\}.$$

As a trivial consequence of Lemma 1 we know $S(x) \subseteq S(F(x))$. Then, since $S(x) \subseteq E$, there exists τ such that $S(F^L(x)) = S(F^\tau(x))$, for every $L \geqslant \tau$.

We say that a sequence $x, F(x), \ldots, F^l(x)$, obtained by the synchronous application of the local rule F, has transient length $\tau(x)$ and period $T(x)$ when

$$\forall\ 0 \leqslant t,\ t' < \tau(x) + T(x),\ t \neq t', \quad F^t(x) \neq F^{t'}(x)$$

and

$$F^{\tau(x)+T(x)}(x) = F^{\tau(x)}(x).$$

Our first result characterizes the fixed points (i.e. configurations which are invariant under the application of F) for the CAN beginning with a continuous configuration.

Theorem 1. *Let x be continuous, then*

$$T(x) = 1 \quad \textit{if and only if for any } r \in \{1, \ldots, R\},\ J(\gamma_r, x) = 0.$$

Proof. ($\Rightarrow$) Suppose $J(\gamma_r, x) = 0$, for every $r \in \{1, \ldots, R\}$. Then, we get

$$J(C,x) = J(O(C),x) + \sum_{r=1}^{R} n_r(C) J(\gamma_r, x) = J(O(C),x).$$

Since $O(C)$ is an open path, for $L \geqslant |V|$ we obtain $a(i,L,x) = a(i,|V|,x)$. When $a(i,L,x) > 0$ we apply Proposition 2(b) to obtain

$$F_i^L(x) = s^{a(i,L,x)}(x_i) = s^{a(i,|V|,x)}(x_i) = F_i^{|V|}(x) \quad \text{for } i \in V,\ L \geqslant |V|.$$

If $a(i,|V|,x) \leqslant 0$ we know from Proposition 2(a) that $F_i^L(x) = x_i$, $L \geqslant 0$.

($\Leftarrow$) Suppose $T(x) = 1$, i.e. there exists τ, such that

$$F_i^l(x) = F_i^\tau(x) \quad \forall l \geqslant \tau,\ \forall i \in V.$$

We show in this case that $\forall i,j \in V$, $F_i^\tau(x) = F_j^\tau(x)$. Assume $i,j \in V$ such that $F_i^\tau(x) \neq F_j^\tau(x)$. Let $C = (i, i_2, \ldots, j)$ be a path connecting vertices i and j. Let i_k be the first node in C with $F_{i_k}^\tau(x) \neq F_i^\tau(x)$, hence

$$\sigma(F_{i_{k-1}}^\tau(x), F_{i_k}^\tau(x)) \neq 0.$$

Therefore, either $F_{i_{k-1}}^\tau(x) \neq F_{i_{k-1}}^{\tau+1}(x)$ or $F_{i_k}^\tau(x) \neq F_{i_k}^{\tau+1}(x)$ which contradicts $T(x) = 1$. Since

$$\forall i,j \in V, \quad F_i^\tau(x) = F_j^\tau(x) \quad \text{we get } J(\gamma_r, F^\tau(x)) = 0$$

for an arbitrary simple circuit γ_r. Since x is continuous, we know $J(\gamma_r, F^\tau(x)) = J(\gamma_r, x)$. $\square$

Notice that the previous theorem says that the fixed points consist of configurations where all the sites are in the same state.

Corollary 3. *Let G be a tree, then $T(x) = 1$ for every initial configuration x.*

Proof. The skeleton is a subgraph of G then, it is a forest (i.e. each connected component is a tree). So, in the stable skeleton there is no circuit and the right condition in Theorem 1 holds. Then $T(x) = 1$. $\square$

The next result groups some simple relations among the number of states q, the global efficiency $e(x)$ and the period $T(x)$.

Proposition 3. *Let $x \in \mathscr{C}$ and $\bar{s} = \max\{s : \exists \gamma_r,\ L(\gamma_r) = s\}$. Then $q > \bar{s}$ implies $T(x) = 1$. Moreover, $e(x) = 1$ implies $T(x)|q$.*

Proof. We know that for a simple circuit $J(\gamma_r, x) = 0 \bmod q$. Since $J(\gamma_r, x) \leqslant L(\gamma_r) < q$ we conclude $J(\gamma_r, x) = 0$. Therefore $T(x) = 1$.

Now, $e(x) = 1$ implies that, any $\gamma_r = (i, i_2, \ldots, i_{n-1}, i)$ whose efficiency is the global one, satisfies $\sigma(x_{i_{k-1}}, x_{i_k}) = 1$ for $2 \leqslant k \leqslant n$. Since any path $C \in G(i, L)$, for L large enough, visits a global efficient circuit, we can extend C to another path C^{+q} by adding q sites in a global efficient circuit. This path jumps $a(i, L, x) + q$ and since $a(i, L + q, x) \leqslant a(i, L, x) + q$ we conclude $a(i, L + q, x) = a(i, L, x) \bmod q$ for any $i \in V$ and $L \geqslant 5|V|^5/q^2$. Thus, from Proposition 2, $T(x)|q$. $\square$

We say that the sequence $x_i, F_i(x), \ldots, F_i^t(x)$, obtained by looking at the evolution in site i has transient length $\tau_i(x)$ and period $T_i(x)$ when

$$\forall\ 0 \leqslant t,\ t' < \tau_i(x) + T_i(x),\quad t \neq t', \quad F_i^t(x) \neq F_i^{t'}(x)$$

and

$$F_i^{\tau_i(x) + T_i(x)}(x) = F_i^{\tau_i(x)}(x).$$

Theorem 2. *Let $x \in \mathscr{C}$ and let γ_r be a global efficient circuit. Then the period of any site i in γ_r $T_i(x)$, divides $L(\gamma_r)$.*

Proof. Let $L \geqslant 5|V|^4/q$ and $C \in G(i, L)$ with $J(C, x) = a(i, L, x)$. From Lemma 8 we can suppose that C visits γ_r. Then, we define C^1 as the path deduced from C by visiting one more time γ_r. Then, $L(C^1) = L(C) + L(\gamma_r)$ and $J(C^1, x) = J(C, x) + J(\gamma_r, x)$. Therefore,

$$a(i, L + L(\gamma_r), x) \geqslant J(C^1, x) = a(i, L, x) + J(\gamma_r, x).$$

Let $\beta \in G(i, L + L(\gamma_r))$ with $J(\beta, x) = a(i, L + L(\gamma_r), x)$. From Lemma 8 we can suppose that β visits γ_r. From β we build β^- by suppressing one loop into γ_r. Then

$$L(\beta^-) = L(\beta) - L(\gamma_r) \text{ and } \beta^- \in G(i, L).$$

Moreover, $J(\beta^-, x) = J(\beta, x) - J(\gamma_r, x)$, so

$$a(i, L, x) \geqslant J(\beta^-, x) = J(\beta, x) - J(\gamma_r, x) = a(i, L + L(\gamma_r), x) - J(\gamma_r, x) \geqslant a(i, L, x).$$

So, $a(i,L,x) = a(i,L+L(\gamma_r),x) \bmod q$, then from Proposition 2 we get

$$F_i^{L+L(\gamma_r)}(x) = s^{a(i,L+L(\gamma_r),x)}(x_i) = s^{a(i,L,x)}(x_i) = F_i^L(x), \quad L \geqslant \frac{5|V|^4}{q}. \qquad \square$$

Theorem 3. *Let* $x \in \mathscr{C}$ *and* $T(x) > 1$. *Then* $T(x)$ *divides*

$$lcm\{s' : \exists r,\ \gamma_r \text{ is a global efficient circuit with length } s'\}.$$

Proof. Let $L \geqslant 5|V|^5/q^2$ and $C \in G(i,L)$ with $J(C,x) = a(i,L,x)$. Let s the largest length of an efficient circuit for C. We shall prove that $a(i,L+s\cdot u_s,x) = a(i,L,x) \bmod q$, where $u_s = l_s/b$,

$$l_s = lcm\{s' \text{: there exists a global efficient circuit of length } s', s' \neq s\}$$

and

$$b = lcd\{s' \text{: there exists a global efficient circuit of length } s'\}.$$

We build $C^{u_s} \in G(i,L+s\cdot u_s)$ as the extension of C which visits u_s more times one efficient circuit for C of length s, γ^s. Clearly, we have

$$L(C^{u_s}) = L(C_{a,s}) + u_s L(\gamma^s) = L(C) + s\cdot u_s.$$

Moreover,

$$J(C^{u_s},x) = J(C_{a,s},x) + u_s J(\gamma^s,x) = a(i,L,x) + u_s J(\gamma^s,x).$$

Therefore, $a(i,L+s\cdot u_s,x) \geqslant a(i,L,x) + u_s J(\gamma^s,x)$.

Let $\beta \in G(i,L+s\cdot u_s)$ with $J(\beta,x) = a(i,L+s\cdot u_s,x)$. Let v be the largest length of efficient circuits for β and γ^v a such efficient circuit. Consider the accumulated form of β and v $\beta_{a,v}$, accumulating in circuit γ^v. From $\beta_{a,v}$ we build β^{-u_s} by removing $u_s(s/v)$ tours into γ^v.

It can be proved that $u_s(s/v) \in \mathbb{N}$. From Lemma 9 we know that $n_v \cdot v \geqslant L + s\cdot u_s - 5|V|^4/q$. So, β^{-u_s} is well defined. Moreover, its jump and its length are given by

$$L(\beta^{-u_s}) = L(\beta_{a,v}) - u_s\frac{s}{v}v \leqslant L + s\cdot u_s - s\cdot u_s = L$$

and

$$J(\beta^{-u_s},x) = J(\beta_{a,v},x) - u_s\frac{s}{v}J(\gamma^v,x).$$

Since $L \geqslant 4|V|^5/q^2$, from Lemma 7 we get

$$e(x) = \frac{J(\gamma^v,x)}{v} = \frac{J(\gamma^s,x)}{s}.$$

So,

$$J(\beta^{-u_s},x) = J(\beta,x) - u_s J(\gamma^s,x) = a(i,L+s\cdot u_s,x) - u_s J(\gamma^s,x) \leqslant a(i,L,x),$$

which implies $a(i,L+s\cdot u_s,x) = u_s J(\gamma^s,x) + a(i,L,x)$.

Observe that $u = lcm\{s'$: there exists a global efficient circuit of length $s'\} = s \cdot u_s$ because

$$u = lcm(l_s, s) = \frac{s \cdot l_s}{lcd(l_s, s)} = \frac{s \cdot l_s}{b_s} = s \cdot u_s$$

and u does not depend on i, L and C.

Since $x \in \mathscr{C}$, $J(\gamma^s, x) \bmod q = 0$. Then, for every $i \in V$

$$F_i^{L+u}(x) = s^{a_i(L+u,x)}(x_i) = s^{a_i(L,x)}(x_i) = F_i^L(x)$$

for every $L \geqslant 5|V|^5/q^2$. Since this lower bound is independent of i we conclude that $T(x)|u$. □

Corollary 4. *The length of the transient time is bounded by* $5|V|^5/q^2$.

Proof. Direct from the proof of Theorem 3. □

Observe that $T(x)$ is bounded by $\prod_{i=1}^{p(|V|)} |V|$ where $p(m)$ is the number of prime numbers smaller than m. Hence, (see [9, Theorems 6 and 420]).

$$T(x) \leqslant |V|^{p(|V|)} = \mathrm{e}^{p(|V|)\ln|V|} = \mathrm{e}^{\Theta(|V|)}.$$

For the general case (i.e. non-necessarily continuous configurations) we have the following results.

Theorem 4. *For any configuration* x, $T(x)|U = lcm\{s : s = 3, \ldots, |V|\}$.

Proof. It is clear that in each connected component ρ of the stable skeleton the period $T^\rho(x)$ divides $lcm\{s$: there exists a global efficient circuit of length $s\}$, which divides U. Moreover, $T(x)$ divides

$$lcm\{T^\rho(x)\text{: } \rho \text{ a connected component}\},$$

so $T(x)$ divides U. □

Corollary 5. *The transient length for any initial configuration is bounded by* $(5|V|^5/q^2)|E|$.

Proof. From Corollary 4 we know that if the periodic regime has not been reached, each $5|V|^5/q^2$ steps, the skeleton must add at least one edge. But this fact can occur only $|E|$ times, which allows us to conclude. □

The previous period's bound is attained by the CAN exhibited in Fig. 9.

Let x be the following initial configuration for the CAN described in Fig. 9:

$$x_{(i,\cdot)} = 0012(0112)^{i-1}\ i \geqslant 2, \qquad x_{(1,\cdot)} = 0012, \qquad x_{(0,0)} = 2,$$

where $x_{(i,j)}$ denotes the state for the jth site in the ith ring.

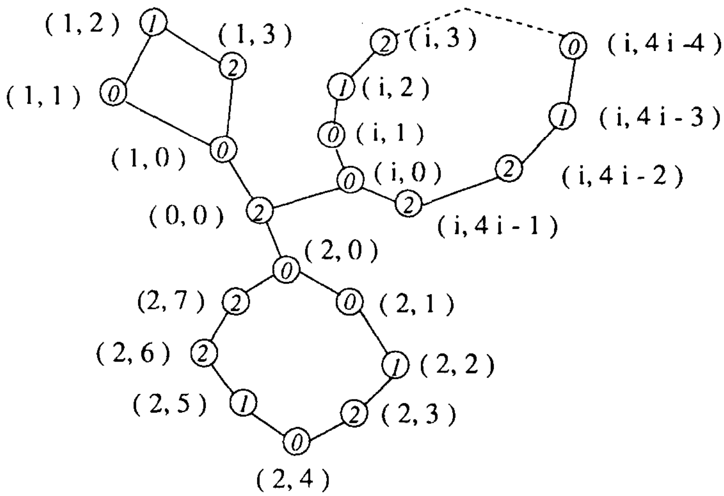


Fig. 9. CAN reaching the bound given in Theorem 3. The graph is composed by $\sum_{i=1}^{m} 4i+1$ sites grouped in rings of size $4i$ connected by a central node $n_c = (0,0)$.

We have the following evolution:

$$x_{(i,\cdot)}^{4k} = (0112)^{i-k}(0012)(0112)^{k-1},\ i \geqslant 2,\ x_{(1,\cdot)}^{4k} = 0012$$

and

$$x_{(0,0)}^{4k} = 2 \text{ for } k \geqslant 1$$

$$x_{(i,\cdot)}^{4k+1} = 112(0112)^{i-1-k}(0012)(0112)^{k-1}0,\ i \geqslant 2,\ x_{(1,\cdot)}^{4k+1} = 0120$$

and

$$x_{(0,0)}^{4k+1} = 0$$

$$x_{(i,\cdot)}^{4k+2} = 12(0112)^{i-1-k}(0012)(0112)^{k-1}01,\ i \geqslant 2,\ x_{(1,\cdot)}^{4k+2} = 1200$$

and

$$x_{(0,0)}^{4k+2} = 1$$

$$x_{(i,\cdot)}^{4k+3} = 2(0112)^{i-1-k}(0012)(0112)^{k-1}012,\ i \geqslant 2,\ x_{(1,\cdot)}^{4k+3} = 2001$$

and

$$x_{(0,0)}^{4k+2} = 1$$

$$x_{(i,\cdot)}^{4k+4} = (0112)^{i-1-k}(0012)(0112)^{k},\ i \geqslant 2,\ x_{(1,\cdot)}^{4k+4} = 0012$$

and

$$x_{(0,0)}^{4k+4} = 2$$

So, each ring evolves independently and the pattern in the central site, $(0,0)$, will be $0012(0112)^u$ where $u = lcm\{4i : i = 1,\dots,m\}$. Observe that this quantity is bounded from below by $\prod_{i=1}^{p(m)} p_i$, where p_i is the ith prime number. Since $p(m) = \Theta(m/\log(m))$ and $\sum_{i=1}^{p(m)} \log(p_i) = \Theta(p(m)\log(m))$ (see [9]) we obtain that the period is at least

$$e^{\sum_{i=1}^{p(m)} \log(p_i)} = e^{\Theta(m)}.$$

Since $m = \Theta(\sqrt{|V|})$, we conclude that $T(x) = e^{\Omega(\sqrt{|V|})}$ which is a non-polynomial bound.

5. Conclusion

In this paper we have studied the principal aspect of the CAN's dynamical behavior: its transient time and the period lengths. To study these aspects we have introduced some mathematical tools which allow us to characterize the evolution essentially in terms of the graph structure: continuity, firing paths, jumps and efficiency of circuits.

Moreover, these tools may be applied for particular classes of graph, for instance the two-dimensional lattice which is the usual cellular space to model excitable automata.

Some other related problems that may be studied in the previous framework are characterizing the periodic behavior in some particular graphs (here we characterize this aspect for trees). A possible generalization could be to define instead of a state ring, a contact process where the state dynamics is driven by a finite machine (Q, Γ) where Q is the state set and $\Gamma \subseteq Q \times Q$ the admissible transitions. Clearly, the previous ring of states is a particular case. Unfortunately, in this general context the important notions of continuity, jumps and efficiency do not necessarily hold.

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